

Energetic cost of superadiabatic quantum computation

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ABSTRACT

We discuss the energetic cost of superadiabatic models of quantum computation. Specifically, we investigate the energy-time complementarity in general transitionless controlled evolutions and in shortcuts to the adiabatic quantum search over an unstructured list. We show that the additional energy resources required by superadiabaticity for arbitrary controlled evolutions can be minimized by using probabilistic dynamics, so that the optimal success probability is fixed by the choice of the evolution time. In the case of analog quantum search, we show that the superadiabatic approach induces a non-oracular counter-diabatic Hamiltonian, with the same energy-time complexity as equivalent adiabatic implementations.

Keywords: Quantum Computing, Quantum Information, Shortcuts to Adiabaticity, Superadiabaticity, Quantum Gates, Quantum Search

1 INTRODUCTION

Shortcuts to adiabatic passage [1, 2, 3, 4] provide a remarkable mechanism for speeding up quantum tasks, which can be achieved through the use of a counter-diabatic assistant driving. These techniques have been introduced to mimic the transitionless adiabatic dynamics, but with the usual constraint on the adiabatic runtime lifted. Transitionless quantum driving has been applied to a number of quantum information protocols, such as population transfer [5, 6] and entanglement generation [7, 8, 9, 10]. In the context of many-body systems, realizable settings have been investigated for assisted evolutions in quantum critical phenomena [11, 12, 13]. More recently, counter-diabatic approaches have been proposed for fast implementation of individual unitaries in quantum circuits, leading to universal *superadiabatic* schemes of quantum computing (QC) via local Hamiltonians [14, 15]. Such methods may be potentially relevant to accelerating the implementation of n -qubit controlled gates in digitized proposals of adiabatic quantum computing (see, e.g., Refs. [16, 17, 18, 19]).

The superadiabatic speedup is intrinsically connected with an increase of the energy resources demanded by the quantum computer [14, 15], which in turn implies a rather versatile computational cost that is controlled by the energetic capacity available to the physical apparatus. Here we show that this energy-time complementarity can be exploited in quantum information processing. First, we consider controlled evolutions (CE) as a mechanism to implement superadiabatic universal QC [14] which generalizes the original adiabatic approach introduced in Ref. [16]. We then show that, within the superadiabatic scenario, the energetic cost can be minimized by replacing the deterministic realization of quantum gates for probabilistic implementations based on a probability distribution of a binary random variable described by an angle parameter. By doing so, the energy expense can be minimized by adjusting the probability distribution, provided the choice of the evolution time of the computational process. Second, we analyze the effects of the energy-time complementarity in analog quantum search [20], where the oracular approach designed by the local adiabatic Grover algorithm is known to be optimal [21, 22]. In this case, we show that the superadiabatic approach naturally requires an unphysical non-oracular counter-diabatic Hamiltonian, with the energy-time complexity equivalent to non-oracular adiabatic implementations.

The paper is organized as follows. In Section 2, we describe the adiabatic implementation of quantum gates via CE and several adiabatic quantum search approaches. We then provide their superadiabatic versions and introduce the metric for energetic cost used in our work. In Section 3, we investigate the energy complexity of the superadiabatic realizations of both quantum gates via CE and analog quantum search. In particular, we consider the properties of the probabilistic model of QC through CE and the consequences of the energy-time complementarity for the search problem. In Section 4, we present our conclusions and future perspectives.

2 METHODS

Our aim in this Section is to discuss adiabatic implementations of QC, their superadiabatic generalizations, and the energetic cost measure adopted in this work.

2.1 Quantum gates by adiabatic controlled evolutions

Let us begin by using adiabatic CE [16] to implement n -controlled gates [14]. To this end, we will consider the adiabatic evolution of a composite system \mathcal{TA} associated with a Hilbert space $\mathcal{H}_{\mathcal{T}} \otimes \mathcal{H}_{\mathcal{A}}$, where \mathcal{T} denotes a target subsystem containing $n + 1$ qubits and \mathcal{A} denotes an auxiliary subsystem containing a single qubit. We will use the first n qubits of \mathcal{T} as the control register of the n -controlled gate, while the last qubit will play the role of its target register. Then, a rotation of the target qubit of an angle ϕ around a direction \hat{n} in the Bloch sphere will be performed when the state of the control register is $|11 \cdots 1\rangle$. We will adopt here the decimal representation $|11 \cdots 1\rangle \equiv |N - 1\rangle$, with $N = 2^n$. An n -controlled rotation over a single qubit can be adiabatically implemented by preparing the auxiliary qubit in the initial state $|0\rangle$, with the adiabatic Hamiltonian given by [14]

$$H(s) = [\mathbb{1} - P_{N-1,n-}] \otimes H_0(s) + P_{N-1,n-} \otimes H_\phi(s) , \quad (1)$$

where $P_{k,n\pm} = |k\rangle \langle k| \otimes |\hat{n}_\pm\rangle \langle \hat{n}_\pm|$ is the set of all orthogonal projectors on the subspace \mathcal{T} and $|\hat{n}_\pm\rangle \langle \hat{n}_\pm| = 1/2 (\mathbb{1} \pm \hat{n} \cdot \vec{\sigma})$, with $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. The Hamiltonians $H_0(s)$ and $H_\phi(s)$ are given by

$$H_\xi(s) = -\hbar\omega \{ \sigma_z \cos \theta(s) + \sin \theta(s) [\sigma_x \cos \xi + \sigma_y \sin \xi] \} , \quad (2)$$

where $\theta(s) = \theta_0 s$, θ_0 is a constant angle, $\xi = \{0, \phi\}$, and s denotes the normalized time $s = t/\tau$, with τ the total evolution time. The system is prepared in the initial state $|\Psi(0)\rangle = |\psi_n\rangle \otimes |0\rangle$, where

$$|\psi_n\rangle = \sum_{m=0}^{N-1} \sum_{\epsilon=\pm} \gamma_{m,\epsilon} |m, \hat{n}_\epsilon\rangle. \quad (3)$$

Then, by adiabatic evolution, the system will evolve to the final state $|\Psi(1)\rangle$ given by

$$|\Psi(1)\rangle = [(\mathbb{1} - P_{N-1, \hat{n}_-}) |\psi_n\rangle] \otimes |E_0^0(1)\rangle + P_{N-1, \hat{n}_-} |\psi_n\rangle \otimes |E_\phi^0(1)\rangle, \quad (4)$$

where $|E_\xi^0(1)\rangle = \cos(\theta_0/2) |0\rangle + e^{i\xi} \sin(\theta_0/2) |1\rangle$ is the ground state of $H_\xi(1)$. Then, equivalently, we can write

$$|\Psi(1)\rangle = \cos(\theta_0/2) |\psi_n\rangle \otimes |0\rangle + \sin(\theta_0/2) |\psi_n^{\text{rot}}\rangle \otimes |1\rangle, \quad (5)$$

with

$$|\psi_n^{\text{rot}}\rangle = \sum_{k=0}^{N-2} \sum_{\epsilon=\pm} \gamma_{k,\epsilon} |k, \hat{n}_\epsilon\rangle + |N-1\rangle \otimes [\gamma_{N-1,+} |\hat{n}_+\rangle + e^{i\phi} \gamma_{N-1,-} |\hat{n}_-\rangle]. \quad (6)$$

The rotated state $|\psi_n^{\text{rot}}\rangle$ is the target of the n -controlled gate. However, note that $|\Psi(1)\rangle$ in Eq. (5) is an entangled state. Thus a measurement must be performed on the auxiliary system, where the action of the gate will be considered successful if \mathcal{A} is measured in the state $|1\rangle$, which occurs with probability $\sin^2(\theta_0/2)$. On the other hand, if the outcome of a measurement on \mathcal{A} yields $|0\rangle$, the adiabatic evolution should be restarted through the Hamiltonian in Eq. (1), as the state of the system is projected onto the initial state $|\Psi(0)\rangle$. Naturally, by choosing $\theta_0 = \pi$, we deterministically ensure the success of the computation. However, as we will show, deterministic evolutions may demand more energy resources than probabilistic processes when transitionless drivings are considered. In particular, observe also that the scheme presented here allows for the implementation of arbitrary n -controlled gates, which lead to versatile sets of universal gates, e.g., single qubit rotations and controlled-NOT operations [23].

2.2 Adiabatic quantum search

Instead of adiabatic implementations of quantum circuits, we can also consider the original approach of adiabatic QC [24], where a single annealing process is performed using energy penalties attributed to quantum states that violate the solutions of an optimization problem. Here we employ this method to analyze three possible adiabatic implementations of quantum search over an unstructured list. An adiabatic QC approach for the quantum search through Grover's algorithm [20] was first proposed in Ref.[25] and improved by using local adiabaticity [21, 22], where the adiabatic evolution is required for each local time interval, instead of being globally applied as in the original proposal. In both cases, the search for a marked element in an unstructured list of $N = 2^n$ elements (labeled by n qubits) can be achieved by employing a Hamiltonian of the form

$$H_0(s) = f(s)(\mathbb{1} - |+\rangle\langle +|) + g(s)(\mathbb{1} - |m\rangle\langle m|), \quad (7)$$

where $|m\rangle$ is the marked state, s is the normalized time ($0 \leq s \leq 1$), $|+\rangle = 1/\sqrt{N} \sum_{i=0}^{N-1} |i\rangle$, and $f(0) = g(1) = 1$ and $f(1) = g(0) = 0$. The eigenspectrum of this Hamiltonian can be exactly derived (see,

e.g., Ref. [26, 27]). In particular, the two lowest eigenstates can be written as

$$|E_{\pm}(s)\rangle = \mathcal{N}_{\pm}(s) [|m\rangle + b_{\pm}(s)|\phi\rangle], \quad (8)$$

where the normalization constant is $\mathcal{N}_{\pm}(s) = 1/\sqrt{1 + (N-1)b_{\pm}(s)^2}$, $|\phi\rangle = \sum_{i \neq m} |i\rangle$, and

$$b_{\pm}(s) = 1 - \frac{E_{\pm}(s)}{f(s)\bar{N}}, \quad (9)$$

with $\bar{N} = 1 - 1/N$, and the corresponding energies $E_{\pm}(s)$ given by

$$E_{\pm}(s) = \frac{f(s) + g(s) \pm \sqrt{[f(s) + g(s)]^2 - 4f(s)g(s)\bar{N}}}{2}. \quad (10)$$

The other higher-energy eigenstates form an $(N-2)$ -fold degenerate eigenspace, whose energy is given by

$$E_{\text{deg}} = [f(s) + g(s)]. \quad (11)$$

In order to explicitly provide the eigenstates $|E_{\text{deg}}^k\rangle$ ($k = 1, \dots, N-2$) associated with the eigenenergy E_{deg} , we write

$$|E_{\text{deg}}^k\rangle = \sum_{n=0}^{N-1} c_n^k |n\rangle. \quad (12)$$

Then, from the eigenvalue equation for $H_0(s)$, it directly follows that the set $\{c_n^k\}$ is just required to satisfy the constraints $\sum_{n=0}^{N-1} c_n^k = 0$ and $c_m^k = 0$. As a consequence, the states $|E_{\text{deg}}^k\rangle$ can be suitably chosen as *time-independent* vectors.

By imposing a local adiabatic evolution [21, 22], i.e. by requiring adiabaticity at each infinitesimal time interval, the runtime is minimized for the path (see also, e.g., Ref. [17])

$$f(s) = 1 - g(s), \quad g(s) = \frac{\sqrt{N-1} - \tan[\arctan(\sqrt{N-1})(1-2s)]}{2\sqrt{N-1}}. \quad (13)$$

This results in a quadratic speedup over the classical search, i.e., we obtain the time complexity $O(\sqrt{N})$ expected by the Grover quantum search [21, 22].

It is possible to reduce the time complexity of the Grover quantum search by transferring the algorithmic cost to other physical resources. The second implementation of the adiabatic Grover search considered here has been introduced in Ref. [26, 28]. It is also based on the Hamiltonian in Eq. (7) to perform the evolution, but requiring that the functions $f(s)$ and $g(s)$ satisfy

$$f(s) = 1 - s + \sqrt{N}(1-s)s, \quad (14)$$

$$g(s) = s + \sqrt{N}(1-s)s. \quad (15)$$

This implementation achieves the solution at constant time complexity $O(1)$. As is apparent from Eqs. (14) and (15), the original time resource has been transferred to the coupling strengths $f(s)$ and $g(s)$ and as discussed in detail in the next Section, will be reflected in the energy scaling required by the system.

The two previous versions of the adiabatic Grover's algorithm are based on oracular Hamiltonians, which we take here to be operators able to recognize the correct answer of a problem [23]. This is indeed the case if one chooses a Hamiltonian composed of an operator O_m in the form $O_m = \mathbb{1} - |m\rangle\langle m|$. The action of O_m in the computational basis $\{|i\rangle\}$ is

$$O_m|i\rangle = (\mathbb{1} - |m\rangle\langle m|)|i\rangle = \begin{cases} 0 & (i = m), \\ |i\rangle & (i \neq m), \end{cases} \quad (16)$$

so that this operator recognizes the marked state, providing no hint about its identity if acting upon any other state. Adiabatic versions of the quantum search have also been proposed via non-oracular Hamiltonians. Our third implementation of Grover's algorithm is based on the non-linear non-oracular (NLNO) Hamiltonian proposed in Ref. [29]. In this work, the time-dependent Hamiltonian in Eq. (7) is replaced for

$$H_0(s) = f(s)(\mathbb{1} - |+\rangle\langle +|) + g(s)(\mathbb{1} - |m\rangle\langle m|) + h(s)(|+\rangle\langle m| + |m\rangle\langle +|), \quad (17)$$

where $h(0) = h(1) = 0$. The Hamiltonian in Eq. (17) contains an operator $\bar{O}_m = |+\rangle\langle m| + |m\rangle\langle +|$. The action of \bar{O}_m in the computational basis $\{|i\rangle\}$ is

$$\bar{O}_m|i\rangle = (|+\rangle\langle m| + |m\rangle\langle +|)|i\rangle = \begin{cases} \frac{1}{\sqrt{N}}|m\rangle + |+\rangle & (i = m), \\ \frac{1}{\sqrt{N}}|m\rangle & (i \neq m). \end{cases} \quad (18)$$

Observe that Eq. (18) implies that \bar{O}_m cannot *exactly* recognize a marked element, even though it could *effectively* recover the marked state for $N \gg 1$ with a single operation over the uniform superposition provided by the state $|+\rangle$. Naturally, the non-oracular form of the Hamiltonian involves all the individual computational states, requiring therefore much more than the capacity of the Hamiltonian to recognize the marked element. This is an obviously artificial approach, whose discussion here is kept just for comparison with the superadiabatic scenario. Assuming a restricted feasibility of such a Hamiltonian, we proceed by looking at its eigenspectrum. The ground and first excited states have the same structure as in Eq. (8), with

$$b_{\pm}(s) = \frac{\bar{N}f(s) + \frac{2h(s)}{\sqrt{N}} - E_{\pm}(s)}{\bar{N} \left[f(s) - h(s)\sqrt{N} \right]}. \quad (19)$$

The two lowest energy levels are given by

$$E_{\pm}(s) = \frac{1}{2} \left\{ f(s) + g(s) + \frac{2h(s)}{\sqrt{N}} \pm \sqrt{[f(s) + g(s)]^2 - 4f(s)g(s)\bar{N} + 4h^2(s) - \frac{4h(s)}{\sqrt{N}}[f(s) + g(s)]} \right\}. \quad (20)$$

As before, the higher-energy states form an $(N - 2)$ -fold degenerate subspace, with energy given by $f(s) + g(s)$. As shown in Ref. [29], this formulation also shows constant time complexity $O(1)$, which can be obtained by choosing a suitable interpolation, such as

$$f(s) = 1 - s, \quad g(s) = s, \quad h(s) = s(1 - s). \quad (21)$$

2.3 Speeding up adiabaticity through superadiabatic evolutions

The performance of adiabatic QC is dictated by a long total evolution time compared to the inverse of a power of the energy gap [30, 31, 32, 33]. However, the adiabatic evolution can be sped up through shortcuts to adiabaticity via counter-diabatic Hamiltonians [1, 2, 3]. The fundamental idea underlying these shortcuts to adiabaticity is to add a new contribution $H_{CD}(t)$, called *counter-diabatic Hamiltonian*, to the original adiabatic Hamiltonian $H(t)$. This term is constructed such that it allows the mimicking of the adiabatic evolution, however without any constraint on the total time of evolution. The total composite Hamiltonian is

$$H_{SA}(t) = H(t) + H_{CD}(t) , \quad (22)$$

which is called *superadiabatic Hamiltonian*. In particular, it is possible to show that the counter-diabatic term reads [3]

$$H_{CD}(t) = i\hbar \sum_n |\dot{n}(t)\rangle \langle n(t)| + \langle \dot{n}(t)|n(t)\rangle |n(t)\rangle \langle n(t)| , \quad (23)$$

where $|n(t)\rangle$ is the eigenstate of $H(t)$ associated to the energy $E_n(t)$. The goal of the counter-diabatic term $H_{CD}(t)$ in the Hamiltonian $H_{SA}(t)$ is exactly to eliminate the diabatic contributions of $H(t)$. Thus, if the system is initially prepared in the ground state of $H(0)$, then the system will deterministically evolve to the instantaneous ground state of the Hamiltonian $H(t)$ with no constraints over the evolution time. Note that, in general, one would need to be able to explicitly calculate all the eigenstates of $H(t)$ to derive a shortcut to adiabaticity using the counter-diabatic driving. However, this may not be a hard requirement in the case of superadiabatic versions of circuit implementations, where one-qubit rotations and two-qubit entangling gates are enough to achieve QC universality [23]. In particular, as we shall see for this case, $H_{CD}(t)$ can be realized through a simple time-independent operator.

2.4 Energetic cost of quantum evolutions

To quantify the expense of energy in a quantum evolution driven by a Hamiltonian $H(t)$, we adopt as the cost measure the average norm of $H(t)$ computed for a total time of evolution τ . This yields [14, 15, 17, 34]

$$\Sigma(\tau) = \frac{1}{\tau} \int_0^\tau \|H(t)\| dt = \int_0^1 \|H(s)\| ds , \quad (24)$$

where $s = t/\tau$ is the parametrized time and the norm here is defined by the Frobenius norm (Hilbert-Schmidt norm) $\|A\| = \sqrt{\text{Tr}[A^\dagger A]}$. Naturally, other norms can be adopted as, for instance, the spectral norm $\|A\|_2 = \sqrt{\lambda_{\max}[A^\dagger A]}$, where λ_{\max} denotes the maximum eigenvalue of $[A^\dagger A]$. For the Hamiltonians investigated in this work, these norms will imply into a cost simply related by a constant $D^{1/2}$, with D denoting the dimension of corresponding the Hilbert space. The Frobenius norm as well as arbitrary superadiabatic evolutions with total evolution time τ , the energetic cost can be written as

$$\Sigma_{SA}(\tau) = \frac{1}{\tau} \int_0^\tau \sqrt{\text{Tr}[H_{SA}^2(t)]} dt = \frac{1}{\tau} \int_0^\tau \sqrt{\text{Tr}[H^2(t) + H_{CD}^2(t)]} dt, \quad (25)$$

where we have used that $\text{Tr}(\{H(t), H_{CD}(t)\}) = 0$ [14]. This explicitly shows that a superadiabatic evolution has an energetic cost larger than its corresponding adiabatic evolution. By evaluating the trace in

Eq. (25), we obtain

$$\Sigma_{\text{SA}}(\tau) = \int_0^1 \sqrt{\sum_m \left[E_m^2(s) + \hbar^2 \frac{\mu_m(s)}{\tau^2} \right]} ds, \quad (26)$$

where $\mu_m(s) = \langle \partial_s m(s) | \partial_s m(s) \rangle - |\langle m(s) | \partial_s m(s) \rangle|^2$ and $\{E_m(s)\}$ is the energy spectrum of the adiabatic Hamiltonian $H(t)$, with $\{|m(s)\rangle\}$ denoting its eigenbasis. Notice that the adiabatic limit is recovered when taking $\tau \rightarrow \infty$. Thus, the speedup obtained by the superadiabatic dynamics is limited by the energetic cost of the evolution. Indeed, this energy-time complementarity can be formally discussed through the quantum speed limit [35], which suggests that the superadiabatic evolution time is compatible with arbitrarily short time intervals (implying into corresponding arbitrarily large energies) [14], while the adiabatic evolution time obeys the lower bound $\tau_{\text{Ad}} \propto 1/\omega^n$, with ω associated with the energy gap and $n \in \mathbb{N}^+$ [30, 31, 32, 33].

3 RESULTS

We now consider the performance of adiabatic and superadiabatic quantum computation, focusing on their time-energy complexity. This will be investigated both for the universal model of superadiabatic controlled gates and for the superadiabatic implementations of the Grover search.

3.1 Quantum gates by superadiabatic controlled evolutions

Let us begin by discussing the superadiabatic model of universal QC via CE implemented by shortcuts to adiabaticity [14]. To this end, let us first write the complete set of eigenstates of $H(t)$ as [14]

$$|E_{0m}^{\epsilon k}(s)\rangle = |m, \hat{n}_\epsilon\rangle \otimes |E_0^k(s)\rangle, \quad (27)$$

$$|E_{0(N-1)}^{+k}(s)\rangle = |N-1, \hat{n}_+\rangle \otimes |E_0^k(s)\rangle, \quad (28)$$

$$|E_{\phi(N-1)}^{-k}(s)\rangle = |N-1, \hat{n}_-\rangle \otimes |E_\phi^k(s)\rangle, \quad (29)$$

where $m \in \{0, \dots, N-2\}$, $\epsilon, k \in \{\pm\}$, and

$$|E_\xi^+(s)\rangle = -\sin \frac{\theta_0 s}{2} |0\rangle + e^{i\xi} \cos \frac{\theta_0 s}{2} |1\rangle, \quad (30)$$

$$|E_\xi^-(s)\rangle = \cos \frac{\theta_0 s}{2} |0\rangle + e^{i\xi} \sin \frac{\theta_0 s}{2} |1\rangle, \quad (31)$$

with $\xi \in \{0, \phi\}$ and $\{|E_\xi^\pm(s)\rangle\}$ denoting the set of eigenstates of each adiabatic Hamiltonian $H_\xi(s)$, as provided by Eq. (2). Thus, by using the Eq. (22), we can show that the superadiabatic Hamiltonian is given by [14]

$$H_{\text{SA}}(s) = [1 - P_{N-1, \hat{n}_-}] \otimes H_0^{\text{SA}}(s) + P_{N-1, \hat{n}_-} \otimes H_\phi^{\text{SA}}(s), \quad (32)$$

where each term $H_\xi^{\text{SA}}(s)$ corresponds to the superadiabatic Hamiltonian associated with the adiabatic Hamiltonian $H_\xi(s)$, i.e. $H_\xi^{\text{SA}}(s) = H_\xi(s) + H_\xi^{\text{CD}}$, with

$$H_\xi^{\text{CD}} = \hbar \frac{\theta_0}{2\tau} (\sigma_y \cos \xi - \sigma_x \sin \xi) \quad (33)$$

being the (time-independent) counter-diabatic contribution to achieve the evolution at total time τ [14].

3.2 Energy-time complementarity in the CE model of quantum gates

Let us now consider Eq. (26) to investigate the time-energy complementarity relationship in both adiabatic and superadiabatic CE models of universal quantum gates. To this end, we need the set of eigenvalues and eigenstates of the adiabatic Hamiltonian in Eq. (1), which are given by Eqs. (27)-(31). The spectrum of $H(s)$ has $(2N)$ -degenerate levels, with $\{|E_{0m}^{\epsilon+}(s)\rangle, |E_{0(N-1)}^{++}(s)\rangle, |E_{\phi(N-1)}^{-+}(s)\rangle\}$ and $\{|E_{0m}^{\epsilon-}(s)\rangle, |E_{0(N-1)}^{+-}(s)\rangle, |E_{\phi(N-1)}^{--}(s)\rangle\}$ associated with the levels $E^+ = \hbar\omega$ and $E^- = -\hbar\omega$, respectively. So, by using Eqs. (27)-(31), we can show that $\mu_l^m(s) = \theta_0^2/4\tau^2$. In addition, the energetic cost to implement any gate controlled by n qubits is $\Sigma_{SA}(\tau, n) = 2^{n/2}\Sigma_{SA}^{sing}(\tau)$ [14], where Σ_{SA}^{sing} is the energetic cost to implement any single qubit unitary transformation, with

$$\Sigma_{SA}^{sing}(\omega\tau, \theta_0) = 2\hbar\omega \sqrt{1 + \frac{\theta_0^2}{4(\omega\tau)^2}}. \quad (34)$$

A similar result can be obtained from the spectral norm, with energetic cost given by $\Sigma_{SA}^{sing}(\omega\tau, \theta_0)|_2 = (1/2)\Sigma_{SA}^{sing}(\omega\tau, \theta_0)$, since the Hilbert space has dimension $D = 4$ in this case. Note that the energetic cost is independent of the parameter θ_0 in the adiabatic limit $\omega\tau \rightarrow \infty$. Therefore, the best computational adiabatic strategy is to set $\theta = \pi$, which deterministically ensures the implementation of the gate with probability one. On the other hand, probabilistic quantum computation can be energetically favored in the superadiabatic regime. Indeed, from Eq. (5), we can see that, by setting $0 < \theta_0 < \pi$, the implementation of the quantum gate is achieved with a nonvanishing probability. Thus, we can investigate whether or not it is possible to find out a specific value of θ_0 such that the energetic cost is better in average than the deterministic choice $\theta_0 = \pi$. To address this point, let us define the quantity

$$\langle N \rangle = \frac{1}{\sin^2(\theta_0/2)}, \quad (35)$$

which is the average number of evolutions for a successful computation. So, the average energetic cost to implement a probabilistic evolution is

$$\bar{\Sigma} = \langle N \rangle \Sigma, \quad (36)$$

where Σ is the cost of a single evolution. Without loss of generality we will consider the cost of single gates, since similar arguments apply for the cost of n -qubit controlled gates. So, by performing superadiabatic probabilistic quantum computing, the average energetic cost is given by

$$\bar{\Sigma}_{SA}^{sing}(\omega\tau, \theta_0) = \langle N \rangle \Sigma_{SA}^{sing}(\omega\tau, \theta_0) = 2\hbar\omega \csc^2\left(\frac{\theta_0}{2}\right) \sqrt{1 + \frac{\theta_0^2}{4(\omega\tau)^2}}. \quad (37)$$

The function $\bar{\Sigma}_{SA}^{sing}(\omega\tau, \theta_0) \rightarrow \infty$ as $\theta_0 \rightarrow 0$ and exhibits a minimum in the interval $0 < \theta_0 < \pi$ as a function of $\omega\tau$. Indeed, the angle θ_0^{\min} that minimizes $\bar{\Sigma}_{SA}^{sing}(\omega\tau, \theta_0)$ grows monotonically with $\omega\tau$, with $\theta_0^{\min} \rightarrow \pi$ as $\omega\tau \rightarrow \infty$ (adiabatic limit). Then, optimizing $\bar{\Sigma}_{SA}^{sing}(\omega\tau, \theta_0)$ for θ_0 , we obtain

$$\frac{\partial}{\partial \theta_0} \bar{\Sigma}_{SA}^{sing}(\omega\tau, \theta_0) = \eta(\theta_0, \omega\tau) \left\{ \theta_0 - \left[4(\omega\tau)^2 + \theta_0^2 \right] \cotan \frac{\theta_0}{2} \right\} = 0, \quad (38)$$

where we have defined the function

$$\eta(\theta_0, \omega\tau) = \frac{\csc^2(\theta_0/2)}{2(\omega\tau)^2 \sqrt{1 + \theta_0^2/4(\omega\tau)^2}}. \quad (39)$$

Note that $\eta(\theta_0, \omega\tau)$ is nonvanishing in the whole interval $\mathcal{I}_{\theta_0} \in [0, \pi]$. Thus, to obtain the critical angle θ_0^{\min} in $\bar{\Sigma}_{SA}^{sing}(\omega\tau, \theta_0)$, we use Eq. (38) to note that $\omega\tau$ satisfies

$$\omega\tau = \frac{\sqrt{\theta_0^{\min}}}{2} \sqrt{\tan\left(\frac{\theta_0^{\min}}{2}\right) - \theta_0^{\min}}, \quad (40)$$

where we can see a dependence of θ_0^{\min} on the choice of $\omega\tau$. In addition, note that θ_0^{\min} is such that $\tan\left(\frac{\theta_0^{\min}}{2}\right) \geq \theta_0^{\min}$, since the quantity $\omega\tau$ is required to be real and positive. The probabilistic advantage is plotted in Fig. 1, where it is shown that the optimal value for θ_0 is a continuous function of $\omega\tau$, being distinct of the deterministic implementation $\theta_0 = \pi$. In the inset, we show the global minimum of the average energy for $\omega\tau = 0.01$, which occurs for $\theta_0 < \pi$. In particular, θ_0^{\min} moves away from π as $\omega\tau$ is lowered, i.e., in the strong superadiabatic regime. As $\omega\tau$ shifts towards the adiabatic limit, we find that $\theta_0^{\min} \rightarrow \pi$. The optimization of the energy cost is shown in the lower inset, where we define the fraction of energy required by the optimized probabilistic model as a function of $\omega\tau$ as

$$\Sigma_{\text{rel}}(\omega\tau) = \frac{\bar{\Sigma}_{SA}^{sing}(\omega\tau, \theta_0^{\min})}{\Sigma_{SA}^{sing}(\omega\tau, \pi)}. \quad (41)$$

Notice that $\Sigma_{\text{rel}}(\omega\tau)$ decreases in the superadiabatic regime, implying into a large reduction of the energetic cost for small values of $\omega\tau$. On the other hand, $\Sigma_{\text{rel}}(\omega\tau) \rightarrow 1$ in the adiabatic limit, since $\theta_0^{\min} \rightarrow \pi$.

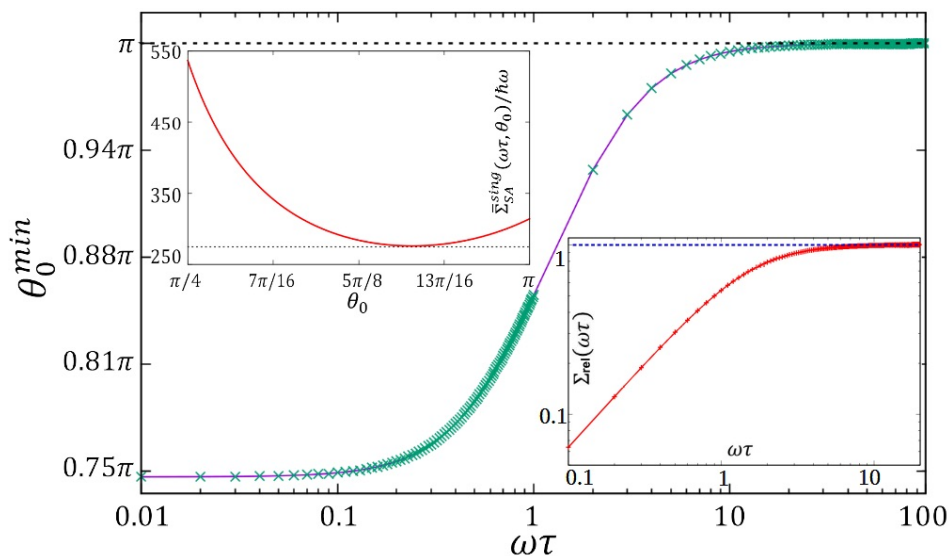


Figure 1. Optimal value θ_0^{\min} for the angle parameter θ_0 as a function of $\omega\tau$, with $\omega\tau$ in logarithmic scale. The points are obtained from Eq. (40), with the curve denoting the numerical fit. **Upper inset:** Average energy in units of $\hbar\omega$ as a function of θ_0 for $\omega\tau = 0.01$. The results are obtained from Eq. (37). **Lower inset:** Fraction $\Sigma_{\text{rel}}(\omega\tau)$ of energy required by the optimized probabilistic model as a function of $\omega\tau$, with data in logarithmic scale. The points are obtained from Eq. (41), with the curve denoting the numerical fit.

3.3 Superadiabatic quantum search

Here, we derive a superadiabatic Hamiltonian $H_{SA}(s)$ for the oracular quantum search governed by the adiabatic Hamiltonian $H_0(s)$ in Eq. (7). We will adopt linear interpolation, with $f(s) = 1 - s$ and $g(s) = s$, as in Ref. [25] and write $H_{SA}(s) = H_0(s) + H_{CD}(s)$. In order to determine the counter-diabatic Hamiltonian $H_{CD}(s)$, we observe that, since $H_0(s)$ has real eigenstates, we use that $\langle \dot{n}(s) | n(s) \rangle = 0$ in Eq. (23), which implies that

$$H_{CD}(s) = \frac{i\hbar}{\tau} \sum_{\xi=\pm} |\dot{E}_\xi(s)\rangle \langle E_\xi(s)|, \quad (42)$$

where the energies $|E_\pm(s)\rangle$ are given by Eq. (8) and

$$|\dot{E}_\pm(s)\rangle = -\frac{(N-1)b_\pm \dot{b}_\pm}{(1+(N-1)b_\pm^2)^{3/2}} |m\rangle + \frac{\dot{b}_\pm}{(1+(N-1)b_\pm^2)^{3/2}} |\phi\rangle. \quad (43)$$

Note that only the ground and first excited states contribute to $H_{CD}(s)$, since the higher energy degenerate sector $\{|E_{\text{deg}}^k\rangle\}$ is composed by time-independent eigenvectors [see Eq. (12)]. Note also that the counter-diabatic Hamiltonian will naturally be non-oracular [see Eq. (18)], with contributions from operators such as $|\phi\rangle\langle m|$ and $|m\rangle\langle \phi|$. This is the reason behind the time complexity $O(1)$ for the superadiabatic Hamiltonian. Naturally, such a result leads to an artificial approach. In a more physical scenario, superadiabaticity could be applied to the quantum search via the direct implementation of the Grover quantum circuit, through the controlled evolution approach discussed in Section 3.1.

3.4 Energy-time complementarity in the quantum search

Let us now analyze the time-energy complementarity relationship in the adiabatic and superadiabatic versions of the Grover search. In the adiabatic regime, the energetic cost can be computed from Eq. (26) and using $\tau \rightarrow \infty$. Therefore, the adiabatic cost can be written as

$$\Sigma_{\text{ad}} = \int_0^1 ds \sqrt{\sum_m [E_m^2(s)]} = \int_0^1 ds \sqrt{E_+(s)^2 + E_-(s)^2 + (N-2)E_{\text{deg}}(s)^2}. \quad (44)$$

Let us initially consider the oracular Hamiltonian $H_0(s)$ in Eq. (7), whose eigenvalues are given by Eqs. (10) and (11). By considering the case of local adiabatic evolution provided by the interpolation in Eq. (13) and by taking $N \gg 1$, we obtain $E_\pm(s) \sim E_{\text{deg}}(s) \sim O(1)$, which implies from Eq. (44) into an energetic cost Σ_{ad}^{LA} that scales as $O(\sqrt{N})$. On the other hand, in the superenergetic version of the quantum search, we adopt the interpolation in Eqs. (14) and (15). Then, by taking $N \gg 1$, we obtain now $E_\pm(s) \sim E_{\text{deg}}(s) \sim O(\sqrt{N})$, which implies into an energetic cost Σ_{ad}^{SE} that scales as $O(N)$. This higher energetic cost is a consequence of the complementarity between energy and time, which arises to compensate the constant time complexity $O(1)$ of the superenergetic version. Naturally, the composite energy-time complexity is kept constant for both cases. This overall complexity is reduced by taking non-oracular artificial Hamiltonians. In the case of the adiabatic NLNO model, we use the Hamiltonian in Eq. (17), whose ground state and first excited state energies are given now by Eq. (20), with the higher energies kept as in Eq. (11). Its energetic cost Σ_{ad}^{NO} can also be computed from Eq. (44) by considering the interpolation in Eq. (21) and by taking $N \gg 1$. Then, we obtain $E_\pm(s) \sim E_{\text{deg}}(s) \sim O(1)$, which yields Σ_{ad}^{NO} scaling as $O(\sqrt{N})$.

For the superadiabatic algorithm, Eq. (26) must be used. Without loss of generality, we set energy units such that $\hbar/\tau = 1$. We find that for $N \gg 1$, the value of $\mu_{\pm}(s)$ in Eq. (26) evaluate to

$$\mu_{\pm}(s) = \langle \dot{E}_{\pm}(s) | \dot{E}_{\pm}(s) \rangle = \frac{(N-1)\dot{b}_{\pm}(s)^2}{(1 + (N-1)b_{\pm}(s)^2)^2}, \quad (45)$$

which in turn gives the superadiabatic search energetic cost Σ_{SA} of order $O(\sqrt{N})$, just reproducing the scaling of the NLNO adiabatic search. Similar results can be obtained if one chooses the spectral norm in the energetic cost, up to a common scaling factor $D^{1/2} = \sqrt{N}$ related to the dimension of the Hilbert space. These results are summarized in Table 2.

	Energy Cost (Frobenius Norm)	Energy Cost (Spectral Norm)	Time Cost
Local adiabatic	$O(\sqrt{N})$	$O(1)$	$O(\sqrt{N})$
Superenergetic	$O(N)$	$O(\sqrt{N})$	$O(1)$
NLNO	$O(\sqrt{N})$	$O(1)$	$O(1)$
Superadiabatic	$O(\sqrt{N})$	$O(1)$	$O(1)$

Table 2. Energy-time complexity for several versions of oracular and non-oracular Hamiltonians for the Grover quantum search.

4 DISCUSSION

We have discussed the energetic cost of shortcuts to adiabaticity and their consequences in quantum information processing. Specifically, we considered both the superadiabatic universal gate model via CE and the superadiabatic analog quantum search. For the gate model, we have shown that, differently from the adiabatic scenario, superadiabatic probabilistic gate implementations are energetically favorable with respect to deterministic gate implementations. This implies that the additional energy resources required by superadiabatic evolutions can be minimized by a suitable probabilistic model. Indeed, probabilistic evolutions have recently been considered in similar applications for QC. In particular, they have been used to cancel errors in adiabatic processes [17] and as a technique to decompose unitary operations [36, 37]. Here, we have shown a new aspect of probabilistic QC, which corresponds to an advantage in the energy balance for superadiabatic dynamics while keeping its performance for a fixed evolution time. For analog quantum search, we have shown that the superadiabatic approach induces a non-oracular counter-diabatic Hamiltonian, with energy-time complexity equivalent to non-oracular adiabatic implementations. This explicitly shows that the Grover optimality is robust against transitionless drivings, which is reflected by a fixed energy-time scaling of the Hamiltonian.

Implications of probabilistic superadiabatic QC under decoherence is a further challenge of immediate interest. In a quantum open-systems scenario, there is a compromise between the time required by adiabaticity and the decoherence time of the quantum device. Therefore, a superadiabatic implementation may provide a direction to obtain an optimal running time for the quantum algorithm while keeping an inherent protection against decoherence. In this context, it is our interest to understand to what extent decoherence can affect the optimal angle θ_0^{\min} , investigating in particular if it can be robust against classes of decohering processes. Concerning specifically the Grover search, it would be interesting to understand

whether superadiabatic implementations are equivalent to arbitrary non-oracular adiabatic Hamiltonians, as suggested in our present discussion. Moreover, the behavior of correlations such as entanglement and the investigation of experimental proposals in the superadiabatic scenario are also topics under investigation.

CONFLICT OF INTEREST STATEMENT

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

AUTHOR CONTRIBUTIONS

All the Authors equally contributed to the conception of the work, development of main the results, and writing of the manuscript.

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